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On C^* -algebras generated by pairs of q -commuting isometries*

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Abstract

We consider the C^* -algebras \mathcal{O}_2^q and \mathcal{A}_2^q generated, respectively, by isometries s_1, s_2 satisfying the relation $s_1^*s_2 = qs_2s_1^*$ with $|q| < 1$ (the deformed Cuntz relation), and by isometries s_1, s_2 satisfying the relation $s_2s_1 = qs_1s_2$ with $|q| = 1$. We show that \mathcal{O}_2^q is isomorphic to the Cuntz–Toeplitz C^* -algebra \mathcal{O}_2^0 for any $|q| < 1$. We further prove that $\mathcal{A}_2^{q_1} \simeq \mathcal{A}_2^{q_2}$ if and only if either $q_1 = q_2$ or $q_1 = \bar{q}_2$. In the second part of our paper, we discuss the complexity of the representation theory of \mathcal{A}_2^q . We show that \mathcal{A}_2^q is $*$ -wild for any q in the circle $|q| = 1$, and hence that \mathcal{A}_2^q is not nuclear for any q in the circle.

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Introduction

The general question of deformations of algebras has received considerable attention in mathematical physics, and in operator algebra theory. The motivation derives from commutation relations of quantum mechanics, see, e.g., [12], and two distinct cases have received special attention. The first case (a) is that of algebras built on rotation models, often called rotation algebras; see [14]. The algebras are generated by a finite set of unitaries, and a multiplicative commutation relation. These generalized rotation algebras are C^* -algebras, and they are labelled by a rotation number, or a rotation matrix, in any case labels depending on continuous parameters. Typically, different parameters yield non-isomorphic C^* -algebras. For example, different irrational rotation numbers yield non-isomorphic simple C^* -algebras. The second case (b) is that of the additive version of the classical quantum commutation relations. This case invites similar deformation questions, but the answers stand in sharp contrast: in case

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(a) we have distinct isomorphism classes, while in case (b) we have isomorphic C^* -algebras for the parameter q in an open set. In [6, 7] the authors showed that the Cuntz algebras, see [3], fit this picture, and that the cases of the canonical commutation relations (CCR) and the canonical anticommutation relations (CAR) arise from this viewpoint as limiting cases. It is shown in [6] that for each n there is a parameter interval J , independent of n , containing 0 but smaller than $-1 < q < 1$, for which a family of C^* -algebras \mathcal{E}_q can be constructed from n generators, so that for $q = 0$ \mathcal{E}_q is \mathcal{O}_n^0 , the Cuntz–Toeplitz algebra, and the family $\{\mathcal{E}_q \mid q \in J\}$ represents only one isomorphism class. The limiting cases $q = -1$ and $q = 1$ correspond to algebras of the CCRs and the CARs, and the question was raised in [6] whether the C^* -algebras \mathcal{E}_q are in fact isomorphic for all q in the ‘natural’ interval, $-1 < q < 1$. This has since become a conjecture of some standing, see, e.g., [10]. We show that there is a closely related deformation family, and we resolve the question for this deformation in the affirmative, in the case $n = 2$.

The relationship between the separate deformation systems is clarified as follows. The deformation of CCR named q_{ij} -CCR was first constructed by Bozejko and Speicher [2]. It is the C^* -algebra generated by $a_i, a_i^*, i = 1, \dots, d$, satisfying the relations

$$a_i^* a_i = 1 + q_i a_i a_i^*, \quad a_i^* a_j = q_{ij} a_j a_i^*, \quad i < j, \quad q_i \in (-1, 1), \quad |q_{ij}| \leq 1.$$

The general conjecture of Jørgensen, Schmitt and Werner [6] states that for any $q_i \in (-1, 1), |q_{ij}| < 1, i, j = 1, \dots, d$, the C^* -algebra generated by these relations is isomorphic to the Cuntz–Toeplitz algebra.

Some limiting cases of the parameters were recently considered by Yuschenko, who showed that the C^* -algebra generated by elements satisfying the relations

$$a_i^* a_i = 1 + q_i a_i a_i^*, \quad q_i \in (0, 1), \quad a_i^* a_j = 0, \quad i \neq j,$$

is isomorphic to the algebra \mathcal{O}_n^0 . We consider another limit situation, putting $q_i = 0$. Then we get the algebras $\mathcal{O}_n^{q_{ij}}$ generated by isometries s_i satisfying relations of the form

$$s_i^* s_j = q_{ij} s_j s_i^*, \quad |q_{ij}| < 1, \quad i < j. \quad (1)$$

The conjecture, in this case, states that for any $|q_{ij}| < 1$ one should get the Cuntz–Toeplitz algebra. In this paper, we give the affirmative answer for the case of two generators. We prove isomorphism of the two C^* -algebras \mathcal{O}_2^q and \mathcal{O}_2^0 for any $|q| < 1$.

A special case of (1) is the case of the algebras generated by q -commuting isometries. It was shown in [13] that the C^* -algebras generated by the generalized quonic relations, see [10], can be generated by isometries satisfying relations of the form

$$s_i^* s_j = q s_j s_i^*, \quad 1 \leq i < j \leq n, \quad |q| = 1. \quad (2)$$

We consider here the situation with $|q| < 1$ which is also the deformation of the Cuntz–Toeplitz algebra. Let us denote the C^* -algebras generated by isometries satisfying (2) by \mathcal{O}_n^q . Then using the general methods developed in [6] one can find some $0 < \varepsilon < 1$ such that for any $|q| < \varepsilon$ one has $\mathcal{O}_n^q \simeq \mathcal{O}_n^0$. And again one has the question whether the isomorphism exists for any q with $|q| < 1$.

In this paper, we show that this is the case for $n = 2$. The methods which we use are different from the general case of $n > 2$, and in any case are of independent interest. The algebras \mathcal{O}_2^q have natural Fock representations, acting on an infinite ‘particle’ Hilbert space, constructed by use of the Fock tensor functor. An important issue, which we resolve, is the faithfulness of the Fock representation.

In section 1, we study the C^* -algebra \mathcal{O}_2^q generated by isometries s_1, s_2 satisfying the deformed Cuntz relation

$$s_1^* s_2 = q s_2 s_1^*, \quad |q| < 1.$$

Indeed, for $q = 0$, we get the Cuntz–Toeplitz C^* -algebra \mathcal{O}_2^0 , see [3], generated by isometries t_1, t_2 satisfying the relation of orthogonality $t_1^*t_2 = 0$.

We prove, see section 1.1, that $\mathcal{O}_2^q \simeq \mathcal{O}_2^0$, C^* -isomorphism, for any $q, |q| < 1$. In section 1.2 we construct special representations of \mathcal{O}_2^q and discuss on an informal level the constructions presented in section 1.1.

The situation with $|q| = 1$ is quite different, see [8, 13, 9]. Almost all \mathcal{O}_2^q with $|q| = 1$ are non-isomorphic. To be more precise, $\mathcal{O}_2^{q_1} \simeq \mathcal{O}_2^{q_2}$ if and only if the corresponding non-commutative tori $\mathcal{A}_{q_j}, j = 1, 2$, are isomorphic. Recall that the non-commutative torus $\mathcal{A}_q, |q| = 1$, is the C^* -algebra generated by a pair of unitaries u_1, u_2 satisfying the relation

$$u_2u_1 = qu_1u_2,$$

see, for example, [14]. Put $q_j = e^{2i\pi\theta_j}, \theta_j \in \mathbb{R}, j = 1, 2$; then $\mathcal{A}_{q_1} \simeq \mathcal{A}_{q_2}$ if and only if $\theta_2 = \pm\theta_1 \pmod{\mathbb{Z}}$.

It is easy to see that the relation $s_1^*s_2 = qs_2s_1^*, |q| = 1$, implies the relation $s_2s_1 = qs_1s_2$. Indeed, for $a = s_2s_1 - qs_1s_2$ one has $a^*a = 0$. However, the converse is not true. In particular, \mathcal{O}_2^q is nuclear for any $q, |q| = 1$, but one of the results of section 2 implies that the C^* -algebra generated by isometries s_1, s_2 satisfying $s_2s_1 = qs_1s_2$ is not nuclear.

In section 2, we consider the C^* -algebra \mathcal{A}_2^q generated by isometries s_1, s_2 satisfying the relation

$$s_2s_1 = qs_1s_2, \quad |q| = 1.$$

The relation $s_2s_1 = qs_1s_2$ implies that $|q| = 1$. Indeed, let the isometries be realized on a Hilbert space \mathcal{H} : then for any non-zero $x \in \mathcal{H}$ one has

$$\|x\| = \|s_2s_1x\| = |q| \|s_1s_2x\| = |q| \|x\|.$$

It is proved, see section 2.1, that $\mathcal{A}_2^{q_1} \simeq \mathcal{A}_2^{q_2}$ if and only if either $q_1 = q_2$ or $q_1 = \bar{q}_2$. Further, in section 2.2 we show that the representation theory of \mathcal{A}_2^q is extremely complicated, even in the case $q = 1$! More precisely, the problem of the classification of the irreducible representations of \mathcal{A}_2^q contains as a sub-problem the description of the irreducible representations of $C^*(\mathcal{F}_2)$, where $C^*(\mathcal{F}_2)$ is the group C^* -algebra of the free group with two generators. The classification of irreducible representations of $C^*(\mathcal{F}_2)$, or equivalently the description of all irreducible pairs of unitary operators, is the standard ‘*-wild’ problem; see [12] for a detailed discussion of the complexity of the representation theory of C^* -algebras.

The properties of pairs of commuting proper isometries were originally studied in [1], where a construction demonstrating the *-wildness of the C^* -algebra \mathcal{A}_2^1 was presented. Note that the construction which we use to prove that \mathcal{A}_2^q is *-wild, is not a generalization of that presented in [1].

1. The C^* -algebra \mathcal{O}_2^q

1.1. The isomorphism $\mathcal{O}_2^q \simeq \mathcal{O}_2^0$

In this section we show that for any $q, |q| < 1$, there is an isomorphism $\mathcal{O}_2^q \simeq \mathcal{O}_2^0$, i.e., isomorphism of C^* -algebras. It is a special case of the hypothesis of Jørgensen, Schmitt and Werner presented in [6, 7]. Namely, it was shown that the C^* -algebras defined by generators $a_i, a_i^*, i = 1, \dots, d$, satisfying the relations

$$a_i^*a_j = \delta_{ij}1 + \sum_{k,l=1}^d T_{ij}^{kl}a_l a_k^*, \quad T_{ij}^{kl} = \overline{T_{ji}^{lk}} \in \mathbb{C},$$

for sufficiently small absolute values of the coefficients, are isomorphic to the Cuntz–Toeplitz C^* -algebra \mathcal{O}_d , see [3], generated by isometries $t_i, i = 1, \dots, d$, satisfying the relations

$$t_i^* t_j = 0, \quad i \neq j.$$

To be more precise, the norm bound $\|T\| < \sqrt{2} - 1$ gives a sufficient condition for this isomorphism. Here T is the self-adjoint operator acting on $\mathcal{H}^{\otimes 2}, \mathcal{H} = \langle e_1, \dots, e_d \rangle$, defined by the action on the basis as follows:

$$T e_k \otimes e_l = \sum_{i,j} T_{ik}^{lj} e_i \otimes e_j.$$

It was conjectured in [7] that the result is correct for $\|T\| < 1$; see also [4].

Below we prove that, for all q in the open interval $-1 < q < 1$, the C^* -algebra \mathcal{O}_2^q can be generated by generators of \mathcal{O}_2^0 , and vice versa.

Remark 1. All of the arguments presented below carry over to the case of any q in the complex disc $|q| < 1$.

We prove our result by several lemmas.

Lemma 1. *Let s_1, s_2 be isometries satisfying the relation $s_1^* s_2 = q s_2 s_1^*$ with $-1 < q < 1$. Construct the elements*

$$\begin{aligned} t_1(s_1, s_2) &= t_1 := s_1, \\ t_2(s_1, s_2) &= t_2 := (1 - s_1 s_1^*) s_2 (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*). \end{aligned}$$

Then $t_i^* t_i = 1, i = 1, 2$, and $t_1^* t_2 = 0$.

Proof. The relation $t_1^* t_2 = 0$ follows from the relation $s_1^* (1 - s_1 s_1^*) = 0$. Let us verify that $t_2^* t_2 = 1$,

$$\begin{aligned} t_2^* t_2 &= (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*) s_2^* (1 - s_1 s_1^*) s_2 (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*) \\ &= (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*) (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*) \\ &\quad - (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*) s_2^* s_1 s_1^* s_2 (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*) \\ &= 1 + \frac{q^2}{1 - q^2} s_1 s_1^* - q^2 (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*) s_1 s_1^* (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*) \\ &= 1 + \frac{q^2}{1 - q^2} s_1 s_1^* - \frac{q^2}{1 - q^2} s_1 s_1^* = 1, \end{aligned}$$

where we used the relation $s_2^* s_1 s_1^* s_2 = q^2 s_1 s_2^* s_2 s_1^* = q^2 s_1 s_1^*$. □

In the following lemma we present the converse construction.

Lemma 2. *Let t_1, t_2 be isometries satisfying $t_1^* t_2 = 0$. Construct the operators*

$$s_1(t_1, t_2) = s_1 := t_1, \quad \tilde{t}_2 := t_2 (1 - t_1 t_1^* + (1 - q^2)^{\frac{1}{2}} t_1 t_1^*)$$

and put

$$s_2(t_1, t_2) = s_2 := \sum_{i=0}^{\infty} q^i t_1^i \tilde{t}_2 (t_1^*)^i, \quad t_1^0 := 1.$$

Then $s_i^* s_i = 1, i = 1, 2$, and $s_1^* s_2 = q s_2 s_1^*$.

Proof. Evidently, for $-1 < q < 1$ the series converges with respect to norm. Let us show that $s_1^*s_2 = qs_2s_1^*$. We will use the obvious relation $t_1^*\tilde{t}_2 = 0$:

$$s_1^*s_2 = t_1^* \sum_{i=0}^{\infty} q^i t_1^i \tilde{t}_2 (t_1^*)^i = \sum_{i=1}^{\infty} q^i t_1^{i-1} \tilde{t}_2 (t_1^*)^i = q \left(\sum_{i=0}^{\infty} q^i t_1^i \tilde{t}_2 (t_1^*)^i \right) t_1^* = qs_2s_1^*.$$

To show that $s_2^*s_2 = 1$ we note firstly that

$$\tilde{t}_2^* \tilde{t}_2 = 1 - q^2 t_1 t_1^*$$

and

$$t_1^i \tilde{t}_2^* (t_1^*)^i t_1^l \tilde{t}_2 (t_1^*)^l = \begin{cases} t_1^i \tilde{t}_2^* (t_1^*)^{i-l} \tilde{t}_2 (t_1^*)^l = 0, & i > l, \\ t_1^i \tilde{t}_2^* t_1^{l-i} \tilde{t}_2 (t_1^*)^l = 0, & l > i, \\ t_1^i (1 - q^2 t_1 t_1^*) (t_1^*)^i, & l = i. \end{cases}$$

Then

$$\begin{aligned} s_2^*s_2 &= \left(\sum_{i=0}^{\infty} q^i t_1^i \tilde{t}_2^* (t_1^*)^i \right) \left(\sum_{l=0}^{\infty} q^l t_1^l \tilde{t}_2 (t_1^*)^l \right) \\ &= \sum_{i=0}^{\infty} q^i t_1^i (1 - q^2 t_1 t_1^*) (t_1^*)^i \\ &= \sum_{i=0}^{\infty} (q^{2i} t_1^i (t_1^*)^i - q^{2i+2} t_1^{i+1} (t_1^*)^{i+1}) = 1. \end{aligned} \quad \square$$

In the following lemma we show that starting from generators of \mathcal{O}_2^q and applying consecutively the constructions presented in lemmas 1 and 2, we get the starting elements.

Lemma 3.

$$s_i(t_1(s_1, s_2), t_2(s_1, s_2)) = s_i, \quad i = 1, 2.$$

Proof. For

$$t_2 = (1 - s_1 s_1^*) s_2 (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*),$$

one has

$$\begin{aligned} \tilde{t}_2 &= (1 - s_1 s_1^*) s_2 (1 - s_1 s_1^* + (1 - q^2)^{-\frac{1}{2}} s_1 s_1^*) (1 - s_1 s_1^* + (1 - q^2)^{\frac{1}{2}} s_1 s_1^*) \\ &= (1 - s_1 s_1^*) s_2. \end{aligned}$$

Note that

$$s_1^i (1 - s_1 s_1^*) s_2 (s_1^*)^i = s_1^i s_2 (s_1^*)^i - q s_1^{i+1} s_2 (s_1^*)^{i+1}.$$

Then we get

$$s_2(t_1, t_2) = \sum_{i=0}^{\infty} q^i s_1^i \tilde{t}_2 (s_1^*)^i = \sum_{i=0}^{\infty} (q^i s_1^i s_2 (s_1^*)^i - q^{i+1} s_1^{i+1} s_2 (s_1^*)^{i+1}) = s_2.$$

In fact we have proved our result. □

Theorem 1. The isomorphism $\mathcal{O}_2^q \simeq \mathcal{O}_2^0$ holds for all q in the open interval $-1 < q < 1$.

Proof. The isomorphism $\phi: \mathcal{O}_2^0 \rightarrow \mathcal{O}_2^q$ is defined by

$$\phi(t_1) = s_1, \quad \phi(t_2) = t_2(s_1, s_2).$$

It follows from lemma 3 that the inverse homomorphism is given by the formulae

$$\psi(s_1) = t_1, \quad \psi(s_2) = s_2(t_1, t_2). \quad \square$$

1.2. Representations of \mathcal{O}_2^q

In this section, we recall the notion of the Fock representation of our q -relations and construct a special class of representations of \mathcal{O}_2^q which includes the Fock one. Using these special representations we discuss informally the results of section 1.1.

Recall that the Fock representation of \mathcal{O}_2^0 (\mathcal{O}_2^q) is the unique irreducible representation defined by the vacuum vector Ω with the property $t_i^*\Omega = 0$ ($s_i^*\Omega = 0$), $i = 1, 2$. It follows from the main result of [5] that the Fock representation of \mathcal{O}_2^q is positive, i.e., it is a $*$ -representation on the Hilbert space, and faithful at the algebraic level. The later means that the Fock representation of the $*$ -algebra generated by the basic relations of \mathcal{O}_2^q has trivial kernel. Our stability theorem implies that the same is at the C^* -algebra level.

Proposition 1. *The Fock representation of \mathcal{O}_2^q is faithful*

Proof. It is known that the Fock representation of the Cuntz–Toeplitz algebra is faithful. This follows from the fact that any irreducible representation of the Cuntz–Toeplitz algebra is either the Fock representation or a representation of the Cuntz algebra (see [3]). Hence we have to show that the Fock representation of \mathcal{O}_2^0 is the Fock representation of \mathcal{O}_2^q , i.e., if $t_i^*\Omega = 0$, $i = 1, 2$, then $s_i^*\Omega = 0$, $i = 1, 2$. Since $s_1 = t_1$ we have to verify only that $s_2^*\Omega = 0$. Indeed, the conditions $t_i^*\Omega = 0$, $i = 1, 2$, imply that $\tilde{t}_2^*\Omega = 0$ and $s_2(t_1, t_2)^*\Omega = 0$. \square

To clarify the nature of the constructions presented in section 1.1, we consider a special class of representations of \mathcal{O}_2^q . Put s_1 to be a multiple of the unilateral shift, i.e., suppose that the representation space is $l_2(\mathbb{N}) \otimes \mathcal{K}$ and that

$$s_1 = S \otimes \mathbf{1} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \mathbf{1} & 0 & 0 & \dots \\ 0 & \mathbf{1} & 0 & \dots \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then it is easy to see that the relation $s_1^*s_2 = qs_2s_1^*$ implies that s_2 has the matrix form

$$s_2 = \begin{pmatrix} u_1 & \sqrt{1-q^2}u_2 & \sqrt{1-q^2}u_3 & \sqrt{1-q^2}u_4 & \dots \\ 0 & qu_1 & q\sqrt{1-q^2}u_2 & q\sqrt{1-q^2}u_3 & \dots \\ 0 & 0 & q^2u_1 & q^2\sqrt{1-q^2}u_2 & \dots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3}$$

where the elements u_i , $i \in \mathbb{N}$, are generators of the Cuntz algebra \mathcal{O}_∞ , i.e., satisfy the relations

$$u_i^*u_i = 1, \quad u_i^*u_j = 0, \quad i \neq j.$$

Analogously, for generators of \mathcal{O}_2^0 we have

$$t_1 = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \mathbf{1} & 0 & 0 & \dots \\ 0 & \mathbf{1} & 0 & \dots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad t_2 = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4}$$

where the elements u_i , $i \in \mathbb{N}$, satisfy the same relations as in the q -deformed case. Then the constructions presented above are just the elementary matrix transformations reducing the matrix of the form (3) to the matrix of the form (4) and vice versa.

Note that the presented motivations cannot be treated as the correct proof of theorem 1 since *a priori* it is not known whether or not the constructed representation of \mathcal{O}_2^q is faithful.

In fact, the construction (4) determines the functor

$$F : \text{Rep } \mathcal{O}_\infty \rightarrow \text{Rep } \mathcal{O}_2^0,$$

(see section 2) defined as follows. For any $\pi \in \text{Ob}(\text{Rep } \mathcal{O}_\infty)$, the representation $F(\pi) \in \text{Ob}(\text{Rep } \mathcal{O}_2^0)$ is constructed by formulae (4) and for any $\Lambda \in \text{Mor}(\pi_1, \pi_2)$ one has $F(\Lambda) = \mathbf{1} \otimes \Lambda$. It can be verified that the functor F is full and faithful. In particular, the representation $F(\pi)$ is irreducible if and only if the representation π is irreducible, and $F(\pi_1)$ is unitarily equivalent to $F(\pi_2)$ if and only if π_1 and π_2 are unitarily equivalent.

Remark 2. It is easy to see that the Fock representation of \mathcal{O}_2^0 corresponds to the Fock representation of \mathcal{O}_∞ .

2. The C^* -algebra \mathcal{A}_2^q

2.1. Description of the isomorphism classes

In this section, we show that $\mathcal{A}_2^{q_1} \simeq \mathcal{A}_2^{q_2}$ if and only if either $q_1 = q_2$ or $q_1 = \bar{q}_2$. In the following, it will be convenient for us to put q in the form $q = e^{2i\pi\theta}$.

Proposition 2. *Let θ be irrational, $q = e^{2i\pi\theta}$. Then there exists a unique normalized tracial state $\tilde{\tau}$ on \mathcal{A}_2^q , and $\mathcal{M} = \{a \mid \tilde{\tau}(a^*a) = 0\}$ is the largest two-sided closed ideal in \mathcal{A}_2^q .*

Proof. Recall that by \mathcal{A}_q we denote the non-commutative torus corresponding to q . Let \mathcal{M} be the closed two-sided ideal generated by the projections $1 - s_j s_j^*$, $j = 1, 2$. Then one has the canonical homomorphism

$$\varphi: \mathcal{A}_2^q \rightarrow \mathcal{A}_2^q / \mathcal{M} = \mathcal{A}_q.$$

Since \mathcal{A}_q with irrational θ is simple, see [14], \mathcal{M} is the largest ideal in \mathcal{A}_2^q . Put $\tilde{\tau} = \tau \circ \varphi$, where τ is the unique normalized tracial state on \mathcal{A}_q , see [14]. Since

$$\mathcal{J} = \{a \mid \tilde{\tau}(a^*a) = 0\}$$

is the two-sided closed ideal and $\tilde{\tau}(1 - s_j s_j^*) = 0$, $j = 1, 2$, one has $\mathcal{M} \subset \mathcal{J}$. Hence $\mathcal{M} = \mathcal{J}$.

Let $\tilde{\sigma}$ be a normalized tracial state on \mathcal{A}_2^q ; then as above,

$$\tilde{\mathcal{M}} := \{a \mid \tilde{\sigma}(a^*a) = 0\} = \mathcal{M},$$

and in particular $\tilde{\sigma}(a) = 0$ for any $a \in \mathcal{M}$. Then one can define the tracial state σ on \mathcal{A}_q by the rule $\sigma(b) = \tilde{\sigma}(a)$, $\varphi(a) = b$. Evidently $\tilde{\sigma} = \sigma \circ \varphi$. By the uniqueness of the trace on \mathcal{A}_q , one has $\tau = \sigma$ and $\tilde{\sigma} = \tilde{\tau}$. □

As a corollary, we have the isomorphism of $\mathcal{A}_2^{q_j}$ with irrational θ_j , $j = 1, 2$.

Proposition 3. *Consider $\mathcal{A}_2^{q_j}$, $q_j = e^{2i\pi\theta_j}$, with irrational θ_j , $j = 1, 2$. Then $\mathcal{A}_2^{q_1} \simeq \mathcal{A}_2^{q_2}$ if and only if $\theta_1 = \pm\theta_2 \pmod{\mathbb{Z}}$.*

Proof. We prove that the isomorphism $\mathcal{A}_2^{q_1} \simeq \mathcal{A}_2^{q_2}$ implies $\theta_1 = \pm\theta_2 \pmod{\mathbb{Z}}$. The converse implication is trivial.

Let $\psi: \mathcal{A}_2^{q_1} \rightarrow \mathcal{A}_2^{q_2}$ be an isomorphism and $\tilde{\tau}_j$, $j = 1, 2$, be normalized traces on $\mathcal{A}_2^{q_j}$, $j = 1, 2$. Denote by $\mathcal{M}_j \subset \mathcal{A}_2^{q_j}$, $j = 1, 2$, the largest ideals introduced in the proposition above.

Consider the normalized tracial state $\widehat{\tau}_1 = \widetilde{\tau}_2 \circ \psi$. Then $\widehat{\tau}_1 = \widetilde{\tau}_1$, and for any $a \in \mathcal{M}_1$ one has

$$\widehat{\tau}_1(a^*a) = \widetilde{\tau}_2(\psi(a^*a)) = 0,$$

i.e., $\psi(a) \in \mathcal{M}_2$ and $\psi(\mathcal{M}_1) \subset \mathcal{M}_2$. Analogously $\psi^{-1}(\mathcal{M}_2) \subset \mathcal{M}_1$. Hence ψ induces the isomorphism

$$\psi: \mathcal{M}_1 \rightarrow \mathcal{M}_2.$$

Denote by ϕ_2 the canonical homomorphism

$$\phi_2: \mathcal{A}_2^{q_2} \rightarrow \mathcal{A}_2^{q_2} / \mathcal{M}_2 = \mathcal{A}_{q_2},$$

then $\ker \phi_2 \circ \psi = \mathcal{M}_1$, hence $\mathcal{A}_{q_1} \simeq \mathcal{A}_{q_2}$ and $\theta_1 = \pm\theta_2 \pmod{\mathbb{Z}}$. □

Let us now consider rational $\theta_j, j = 1, 2$.

Proposition 4. *Let $\theta_j \in \mathbb{Q}, j = 1, 2$. Then $\mathcal{A}_2^{q_1} \simeq \mathcal{A}_2^{q_2}$ if and only if $\theta_1 = \pm\theta_2 \pmod{\mathbb{Z}}$.*

Proof. As in the irrational case, we prove that the existence of an isomorphism

$$\psi: \mathcal{A}_2^{q_1} \rightarrow \mathcal{A}_2^{q_2}$$

implies isomorphism of \mathcal{A}_{q_1} and \mathcal{A}_{q_2} .

Let $\mathcal{M}_j \subset \mathcal{A}_2^{q_j}, j = 1, 2$, be the ideals generated by the projections $1 - s_i s_i^*, i = 1, 2$. We show that $\psi(\mathcal{M}_1) \subset \mathcal{M}_2$. Indeed, consider the canonical homomorphism

$$\varphi_2: \mathcal{A}_2^{q_2} \rightarrow \mathcal{A}_{q_2}$$

and the composite homomorphism

$$\varphi_2 \circ \psi: \mathcal{A}_2^{q_1} \rightarrow \mathcal{A}_{q_2}.$$

Since any irreducible representation of \mathcal{A}_q with rational θ is finite dimensional, \mathcal{A}_q does not contain any non-unitary isometry. Hence $\varphi_2 \circ \psi(1 - s_i s_i^*) = 0$ and $\psi(1 - s_i s_i^*) \in \mathcal{M}_2, i = 1, 2$. So $\psi(\mathcal{M}_1) \subset \mathcal{M}_2$. Analogously, $\psi^{-1}(\mathcal{M}_2) \subset \mathcal{M}_1$. As in the proof of the previous proposition, one has $\mathcal{A}_{q_1} \simeq \mathcal{A}_{q_2}$. It remains only to recall that the rational tori $\mathcal{A}_{q_j}, j = 1, 2$, are isomorphic if and only if $\theta_1 = \pm\theta_2 \pmod{\mathbb{Z}}$. □

2.2. The C^* -algebra \mathcal{A}_2^q is $*$ -wild

In this section, we discuss the complexity of the representation theory of \mathcal{A}_2^q .

To compare C^* -algebras according to the complexity of their categories of representations, we use the relation of majorization. Note that we modify the definition of majorization given in [12] to make it less restrictive.

Recall that the category of representations of a certain C^* -algebra \mathcal{A} , denoted by $\text{Rep } \mathcal{A}$, has the representations of \mathcal{A} as its objects and the intertwining operators as its morphisms.

Definition 1. *We say that a C^* -algebra \mathcal{A} is majorized by a C^* -algebra $\mathcal{B}, \mathcal{A} < \mathcal{B}$, if there exists a homomorphism*

$$\varphi: \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{C},$$

where \mathcal{C} is a nuclear C^* -algebra, and an irreducible representation

$$\widetilde{\pi}: \mathcal{C} \rightarrow B(\mathcal{H})$$

such that the functor

$$\mathcal{F}_\varphi: \text{Rep } \mathcal{A} \rightarrow \text{Rep } \mathcal{B}$$

defined by

$$\begin{aligned} \mathcal{F}_\varphi(\pi) &= (\pi \otimes \tilde{\pi}) \circ \varphi, & \pi &\in \text{Ob}(\text{Rep } \mathcal{A}), \\ \mathcal{F}_\varphi(A) &= A \otimes 1, & A &\in \text{Mor}(\pi_1, \pi_2), \end{aligned}$$

is full and faithful.

Informally, this definition means that using φ one can construct the representations of \mathcal{B} from the representations of \mathcal{A} and the representations of \mathcal{B} are irreducible (unitarily equivalent) if and only if the corresponding representations of \mathcal{A} are irreducible (unitarily equivalent).

It is easy to see that majorization is a partial order.

Definition 2. We say that a C^* -algebra \mathcal{A} is $*$ -wild if

$$C^*(\mathcal{F}_2) \prec \mathcal{A},$$

where $C^*(\mathcal{F}_2)$ is the group C^* -algebra of the free group with two generators.

The group C^* -algebra $C^*(\mathcal{F}_2)$ is considered as the standard $*$ -wild algebra, since it can be shown that the problem of the classification of the representations of $C^*(\mathcal{F}_2)$ contains as a sub-problem the classification of the representations of any finitely generated C^* -algebra, in particular $C^*(\mathcal{F}_n) \prec C^*(\mathcal{F}_2)$; see [12].

Obviously, a C^* -algebra majorizing a $*$ -wild C^* -algebra is $*$ -wild. The $*$ -wild C^* -algebras have a very complicated category of representations: in particular, it was noted in [12] that $*$ -wild algebras are not nuclear. Since our definition of majorization generalizes the one considered in [12], we present below the proof of this statement.

Proposition 5. Let \mathcal{A} be a $*$ -wild C^* -algebra. Then \mathcal{A} is not nuclear.

Proof. It is sufficient to show that \mathcal{A} has a representation generating a non-hyperfinite factor, since any factor representation of a nuclear C^* -algebra is hyperfinite, by a theorem of Alain Connes. Since $C^*(\mathcal{F}_2) \prec \mathcal{A}$, one has the homomorphism

$$\varphi: \mathcal{A}_2^q \rightarrow C^*(\mathcal{F}_2) \otimes \mathcal{C},$$

where \mathcal{C} is a nuclear C^* -algebra, and the irreducible representation

$$\tilde{\pi}: \mathcal{C} \rightarrow B(\mathcal{H}),$$

as in definition 2.2.

Consider a representation π of $C^*(\mathcal{F}_2)$ generating a non-hyperfinite factor. Put $(\pi \otimes \tilde{\pi}) \circ \varphi := \pi_1$ and note that

$$(\pi \otimes \tilde{\pi})(C^*(\mathcal{F}_2) \otimes \mathcal{C})'' = \pi(C^*(\mathcal{F}_2))'' \otimes B(\mathcal{H})$$

is also a non-hyperfinite factor. We use a prime here to denote the commutant (and double primes for the double commutant). Since the functor

$$\mathcal{F}_\varphi: \text{Rep } C^*(\mathcal{F}_2) \rightarrow \text{Rep } \mathcal{A}$$

defined in definition 1 is full and faithful, one has

$$\pi_1(\mathcal{A})' = (\pi \otimes \tilde{\pi})(C^*(\mathcal{F}_2) \otimes \mathcal{C})'.$$

Hence

$$\pi_1(\mathcal{A})'' = (\pi \otimes \tilde{\pi})(C^*(\mathcal{F}_2) \otimes \mathcal{C})''$$

is a non-hyperfinite factor. □

To prove that \mathcal{A}_2^q is $*$ -wild, we need an auxiliary proposition. Note that below we denote by \mathcal{O}_2 the Cuntz algebra, i.e., we suppose that the generators of \mathcal{O}_2^0 satisfy the additional relation $t_1 t_1^* + t_2 t_2^* = 1$.

In the following for any C^* -algebras $\mathcal{A}_i, i = 1, 2$, we denote by $\mathcal{A}_1 \star \mathcal{A}_2$ their free product, see [15].

Proposition 6. *The C^* -algebra $\mathcal{O}_2 \star C([0, 1])$ is $*$ -wild.*

Proof. It was shown in [11] that the C^* -algebra $C([0, 1]) \star C([0, 1])$ is $*$ -wild. Then to prove our statement it is sufficient to show that

$$C([0, 1]) \star C([0, 1]) \prec \mathcal{O}_2 \star C([0, 1]).$$

In the following, we denote by c the standard generator of $C([0, 1]), c(x) = x$, for any $x \in [0, 1]$, and denote by c_1, c_2 the standard free generators of $C([0, 1]) \star C([0, 1])$. Then the needed majorization is given by the homomorphism

$$\varphi: \mathcal{O}_2 \star C([0, 1]) \rightarrow (C([0, 1]) \star C([0, 1])) \otimes \mathcal{O}_2$$

defined by

$$\varphi(x) = 1 \otimes x, \quad x \in \mathcal{O}_2, \quad \varphi(c) = c_1 \otimes t_1 t_1^* + c_2 \otimes t_2 t_2^*$$

and some irreducible representation $\tilde{\pi}: \mathcal{O}_2 \rightarrow B(\mathcal{H})$.

To prove that the induced functor

$$\mathcal{F}_\varphi: \text{Rep } C([0, 1]) \star C([0, 1]) \rightarrow \text{Rep } \mathcal{O}_2 \star C([0, 1])$$

is full and faithful, it is sufficient to show that any $\Lambda \in F(\pi)(\mathcal{O}_2 \star C([0, 1]))'$ has the form $\Lambda_1 \otimes \mathbf{1}$ with $\Lambda_1 \in \pi(C([0, 1]) \star C([0, 1]))'$; see lemma 13 and remark 49 in [12].

Let us put $\mathcal{F}_\varphi(\pi) := \pi_1, \tilde{\pi}(t_i) := T_i$, and $\pi(c_i) = C_i, i = 1, 2$. Then

$$\pi_1(c) = C_1 \otimes T_1 T_1^* + C_2 \otimes T_2 T_2^*, \quad \pi_1(t_i) = \mathbf{1} \otimes T_i, \quad i = 1, 2.$$

Let $\Lambda \in \pi_1(\mathcal{O}_2 \star C([0, 1]))$. Then it is easy to see that the relations

$$\Lambda(\mathbf{1} \otimes T_i) = (\mathbf{1} \otimes T_i)\Lambda, \quad \Lambda(\mathbf{1} \otimes T_i^*) = (\mathbf{1} \otimes T_i^*)\Lambda$$

imply that $\Lambda = \Lambda_1 \otimes \mathbf{1}$. Further, since

$$\pi_1(ct_i t_i^*) = C_i \otimes T_i T_i^*, \quad i = 1, 2,$$

one has

$$\Lambda_1 C_i \otimes T_i T_i^* = C_i \Lambda_1 \otimes T_i T_i^*, \quad i = 1, 2,$$

hence $\Lambda_1 C_i = C_i \Lambda_1, i = 1, 2$, and $\Lambda_1 \in \pi(C([0, 1]) \star C([0, 1]))'$. □

Now we are ready to prove the main result of this part.

Theorem 2. *The C^* -algebra \mathcal{A}_2^q is $*$ -wild.*

Proof. We prove that $\mathcal{O}_2 \star C([0, 1]) \prec \mathcal{A}_2^q$. Let us construct the homomorphism

$$\varphi: \mathcal{A}_2^q \rightarrow (\mathcal{O}_2 \star C([0, 1])) \otimes \mathcal{B},$$

where

$$\mathcal{B} = C^*(s, u \mid s^*s = 1, u^*u = uu^* = 1, us = qsu).$$

This \mathcal{B} is nuclear, since it is the crossed product $\mathcal{B} = T(C(\mathbf{T})) \rtimes \mathbb{Z}$, where $T(C(\mathbf{T}))$ is the Toeplitz C^* -algebra.

Pick a function c taking values strictly between 0 and 1 as a generator of $C([0, 1])$, and let t_1, t_2 be generators of \mathcal{O}_2 . Put

$$a_1 := t_1 c, \quad a_2 := t_2(1 - c).$$

It is easy to verify that the following relations are satisfied:

$$a_1^* a_1 + a_2^* a_2 = 1, \quad a_2^* a_1 = 0. \quad (5)$$

Then we define the images of the generators of \mathcal{A}_2^q as follows:

$$\varphi(s_1) = 1 \otimes s, \quad \varphi(s_2) = a_1 \otimes u + a_2 \otimes su. \quad (6)$$

To verify that $\varphi(s_i^* s_i) = 1$ and $\varphi(s_2 s_1) = q \varphi(s_1 s_2)$, one has only to use relations (5).

Finally, we fix the irreducible representation $\tilde{\pi}$ of \mathcal{B} acting on $\mathcal{K} = l_2(\mathbb{N})$ by

$$\tilde{\pi}(s) = S, \quad \tilde{\pi}(u) = D(q), \quad (7)$$

where

$$S e_n = e_{n+1}, \quad D(q) e_n = q^{n-1} e_n, \quad n \in \mathbb{N}.$$

Then the induced functor

$$\mathcal{F}_\varphi: \text{Rep } \mathcal{O}_2 \star C([0, 1]) \rightarrow \text{Rep } \mathcal{A}_2^q$$

is given by the following construction. Starting with a representation of $\mathcal{O}_2 \star C([0, 1])$, say π , if as above we put $C := \pi(c)$, $T_i := \pi(t_i)$, and $A_i := \pi(a_i)$, $i = 1, 2$, then we define $\mathcal{F}_\varphi(\pi) := \pi_1$ by the formulae

$$\pi_1(s_1) := S_1 = 1 \otimes S, \quad \pi_1(s_2) := S_2 = A_1 \otimes D(q) + A_2 \otimes SD(q). \quad (8)$$

The proof of the fullness and faithfulness of \mathcal{F}_φ is essentially the same as in the proposition above. We note only that the equalities $C^2 = A_1^* A_1$ and $T_i = A_i C^{-1}$, $i = 1, 2$, imply that

$$\{A_i, A_i^*, i = 1, 2\}' = \{T_i, T_i^*, C, i = 1, 2\}' = \pi(\mathcal{O}_2 \star C([0, 1]))',$$

where again the prime denotes the commutant. So one has to show that any $\Lambda \in \pi_1(\mathcal{A}_2^q)'$ has the form $\Lambda = \Lambda_1 \otimes \mathbf{1}$ with $\Lambda_1 \in \{A_i, A_i^*, i = 1, 2\}'$. \square

The following corollary is immediate.

Corollary 1. *The C^* -algebra \mathcal{A}_2^q is not nuclear.*

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